On Monotonic and Orthant Monotonic Norms

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ABSTRACT

This paper gives a characterization of real vector norms with respect to the interval $[-1,1]$.

I. INTRODUCTION

Let $\mathbb{R}^n$ be the n-dimensional vector space of column vectors $x=(x_1,x_2,\dots,x_n)^T$ with real entries x_i, and $e_k\in\mathbb{R}^n$ the vector $(0,\dots,0,1,0,\dots,0)^T$ where the 1 is in the kth position. Let $M_n(\mathbb{R})$ denote the set of $n\times n$ real valued matrices, $I\in M_n(\mathbb{R})$ the identity matrix, and $\mathcal{D}_1\subset M_n(\mathbb{R})$ the set of diagonal matrices $D=\operatorname{diag}(d_1,d_2,\dots,d_n)$ with $\max_k|d_k|=1$. By $\rho(A)$ we denote the spectral radius of A, i.e., $\rho(A)=\max\{|\tau|: \tau \text{ is an eigenvalue of } A\}$. Recall that a vector norm $\|\cdot\|$ on $\mathbb{R}^n$ is a nonnegative real valued function on $\mathbb{R}^n$ satisfying:

$$\|x\|=0 \quad \text{if and only if} \quad x=0, \tag{1.1}$$

$$\|\alpha x\|=|\alpha|\,\|x\| \qquad \text{for} \quad x\in\mathbb{R}^n, \quad \alpha\in\mathbb{R}, \tag{1.2}$$

$$\|x+y\|\leqslant\|x\|+\|y\| \qquad \text{for} \quad x,y\in\mathbb{R}^n. \tag{1.3}$$

The operator norm of a matrix A corresponding to a vector norm $\|\cdot\|$ is

$$\operatorname{lub}(A)=\max_{x\neq 0}\frac{\|Ax\|}{\|x\|}. \tag{1.4}$$

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Since to each eigenvalue of A there corresponds an eigenvector, we immediately obtain the well-known inequality

$$\rho(A) \leqslant \operatorname{lub}(A). \tag{1.5}$$

For any $r \in [-1,1]$ and for any $k = 1,2,\dots,n$, we define the $n \times n$ matrix

$$H_k(r) = I - (r+1)e_k e_k^T. \tag{1.6}$$

If $\mathfrak{N}$ denotes the class of norms $\|\cdot\|$ on $\mathbb{R}^n$, then for any $r \in [0,1] \cup \{-1\}$, the following classes of norms on $\mathbb{R}^n$ are defined:

$$\mathfrak{N}_r = \{\|\cdot\| \in \mathfrak{N} : \operatorname{lub}(H_k(r)) = 1,\ k = 1,2,\dots,n\}. \tag{1.7}$$

Clearly, $\mathfrak{N}_{-1} = \mathfrak{N}$, since for $k = 1,2,\dots,n$, $H_k(-1)$ is the identity matrix and $\operatorname{lub}(I) = 1$ for any $\|\cdot\| \in \mathfrak{N}$. If $x = (x_1, x_2, \dots, x_n)^T$, then $|x| = (|x_1|, |x_2|, \dots, |x_n|)^T$, and for $x, y \in \mathbb{R}^n$ we denote $|x| \leqslant |y|$ if $|x_i| \leqslant |y_i|$ for all $i = 1,2,\dots,n$. A vector norm on $\mathbb{R}^n$ which satisfies one of the following three equivalent conditions is called *monotonic* or *absolute* [1]:

$$\text{if} \quad |x| \leqslant |y| \quad \text{then} \quad \|x\| \leqslant \|y\|, \tag{1.8}$$

$$\||x|\| = \|x\| \qquad \text{for any} \quad x \in \mathbb{R}^n, \tag{1.9}$$

$$\operatorname{lub}(D) = 1 \qquad \text{for any} \quad D \in \mathfrak{D}_1. \tag{1.10}$$

A vector norm $\|\cdot\|$ on $\mathbb{R}^n$ is called an *orthant monotonic norm* if

$$\|x\| \leqslant \|y\| \qquad \text{for} \quad |x| \leqslant |y|$$
$$\text{provided} \quad x_i y_i \geqslant 0 \quad (i = 1,2,\dots,n). \tag{1.11}$$

By using the fact that (1.8) and (1.10) are equivalent, one can similarly prove that the condition (1.11) is equivalent to the following condition:

$$\text{if} \quad D \in \mathfrak{D}_1 \text{ is nonnegative,} \quad \text{then} \quad \operatorname{lub}(D) = 1. \tag{1.12}$$

Funderlic [2] has shown that $\|\cdot\|$ is an orthant monotonic norm if and only if

$$\operatorname{lub}(H_k(0)) = \operatorname{lub}(I - e_k e_k^T) = 1, \qquad k = 1,2,\dots,n. \tag{1.13}$$

In this paper, first we show that for $r \in (-1,0]$ and $k=1,2,\dots,n$,

$$\operatorname{lub}(H_k(r))=1 \quad \text{if and only if} \quad \operatorname{lub}(H_k(0))=1;$$

then we prove that

$$\mathfrak{N}_{r_1} \supset \mathfrak{N}_{r_2} \qquad \text{if} \quad 0 \leqslant r_1 < r_2 \leqslant 1,$$

and we also show that $\mathfrak{N}_1$ is the class of all monotonic norms. This last result is a concrete characterization of monotonic norms on $\mathbb{R}^n$, in the sense that the condition (1.10) may be reduced to the following condition:

$$\operatorname{lub}(H_k(1))=\operatorname{lub}(I-2e_ke_k^T)=1, \qquad k=1,2,\dots,n. \tag{1.14}$$

II. RESULTS

Lemma 2.1. *Let $r \in (-1,0]$. Then the following are equivalent:*

(i) $\operatorname{lub}(H_k(0))=1$, $k=1,2,\dots,n$,
(ii) $\operatorname{lub}(H_k(r))=1$, $k=1,2,\dots,n$.

Proof. If $-1<r_1<r_2\leqslant 0$, then

$$H_k(r_1)=\left(\frac{r_2-r_1}{1+r_2}\right)I+\left(\frac{1+r_1}{1+r_2}\right)H_k(r_2) \qquad (k=1,2,\dots,n);$$

so if $\|\cdot\| \in \mathfrak{N}_{r_2}$, with $\operatorname{lub}(H_k(r_2))=1$ $(k=1,2,\dots,n)$, then

$$\begin{aligned}\operatorname{lub}(H_k(r_1)) &\leqslant \left|\frac{r_2-r_1}{1+r_2}\right|\operatorname{lub}(I)+\left|\frac{1+r_1}{1+r_2}\right|+\operatorname{lub}(H_k(r_2))\\ &=\left(\frac{r_2-r_1}{1+r_2}\right)\cdot 1+\left(\frac{1+r_1}{1+r_2}\right)\cdot 1=1,\end{aligned}$$

and since $\rho(H_k(r_1))=1$, we conclude that $\operatorname{lub}(H_k(r_1))=1$. Thus (i) implies (ii).

Now let $-1<r<0$; then $H_k(0)=\lim_{m\to\infty}(H_k(r_2))$, $k=1,2,\dots,n$. Since $\operatorname{lub}(A)$ is an operator norm, it satisfies

$$\operatorname{lub}(A\cdot B)\leqslant \operatorname{lub}(A)\cdot\operatorname{lub}(B),$$

so for $k=1,2,\ldots,n$, we have

$$\operatorname{lub}(H_k(r)) = \lim_{m\to\infty}\left[\operatorname{lub}([H_k(r)]^m)\right] \leqslant \lim_{m\to\infty}\left[\operatorname{lub}(H_k(r))\right]^m.$$

Now, if $\operatorname{lub}(H_k(r))=1$, then

$$1 \leqslant \operatorname{lub}(H_k(0)) \leqslant \lim_{m\to\infty} 1^m = 1.$$

Therefore $\|\cdot\| \in \mathfrak{N}_0$, i.e., (ii) implies (i). ∎

So the class of all orthant monotonic norms is defined as

$$\mathfrak{N}_0 = \{\|\cdot\| \in \mathfrak{N} : \operatorname{lub}(H_k(r)) = 1, r \in (-1,0], k=1,2,\ldots,n\}.$$

LEMMA 2.2.

$$\mathfrak{N}_{r_1} \supseteq \mathfrak{N}_{r_2} \qquad \text{if} \quad 0 \leqslant r_1 < r_2 \leqslant 1. \tag{1.15}$$

Proof. Let $r_2 \in [0,1]$, then

$$H_k(-r_2) = \left(\frac{2r_2}{1+r_2}\right)I + \left(\frac{1-r_2}{1+r_2}\right)H_k(r_2) \qquad (k=1,2,\ldots,n).$$

Hence, $\|\cdot\| \in \mathfrak{N}_{r_2}$ implies that for all $k=1,2,\ldots,n$,

$$\begin{aligned}\operatorname{lub}(H_k(-r_2)) &\leqslant \left(\frac{2r_2}{1+r_2}\right)\operatorname{lub}(I) + \left(\frac{1-r_2}{1+r_2}\right)\operatorname{lub}(H_k(r_2))\\ &= \left(\frac{2r_2}{1+r_2}\right)\cdot 1 + \left(\frac{1-r_2}{1+r_2}\right)\cdot 1 = 1;\end{aligned}$$

thus $1=\rho(H_k(-r_2)) \leqslant \operatorname{lub}(H_k(-r_2)) \leqslant 1$, so $\operatorname{lub}(H_k(-r))=1$. If $0 \leqslant r_1 < r_2 \leqslant 1$, then

$$H_k(r_1) = \left(\frac{r_2-r_1}{2r_2}\right)H_k(-r_2) + \left(\frac{r_2+r_1}{2r_2}\right)H_k(r_2) \qquad (k=1,2,\ldots,n).$$

Hence, if $\|\cdot\| \in \mathcal{N}_{r_2}$, then for all $k = 1, 2, \ldots, n$,

$$\operatorname{lub}(H_k(r_1)) \leqslant \left|\frac{r_2 - r_1}{2r_2}\right| \operatorname{lub}(H_k(-r_2)) + \left|\frac{r_2 + r_1}{2r_2}\right| \operatorname{lub}(H_k(r_2))$$

$$= \left(\frac{r_2 - r_1}{2r_2}\right) + \left(\frac{r_2 + r_1}{2r_2}\right) = 1.$$

Therefore $\operatorname{lub}(H_k(r_1)) = \rho(H_k(r_1)) = 1$, i.e., $\|\cdot\| \in \mathcal{N}_{r_1}$. ■

Later we show that the inclusions are strict.

THEOREM 2.3. *The class of monotonic norms on* $\mathbb{R}^n$ *consists of the class* $\mathcal{N}_1$.

Proof. If $\|\cdot\|$ is a monotonic norm on $\mathbb{R}^n$, then $\|\cdot\|$ satisfies (1.10); this clearly implies (1.15), i.e., $\operatorname{lub}(I - 2e_k e_k^T) = \operatorname{lub}(H_k(1)) = 1$ $(k = 1, 2, \ldots, n)$.

Now let $\|\cdot\| \in \mathcal{N}_1$ and let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be in $\mathcal{D}_1$; then we can write $D = \prod_{k=1}^n D_k$, where

$$D_k = \left(\frac{1 + d_k}{2}\right) I - \left(\frac{1 - d_k}{2}\right) H_k(1), \qquad k = 1, 2, \ldots, n.$$

Since for all $k = 1, 2, \ldots, n$,

$$1 \leqslant \operatorname{lub}(D_k) \leqslant \left(\frac{1 + d_k}{2}\right) \operatorname{lub}(I) + \left(\frac{1 - d_k}{2}\right) \operatorname{lub}(H_k(1))$$

$$= \left(\frac{1 + d_k}{2}\right) \cdot 1 + \left(\frac{1 - d_k}{2}\right) \cdot 1 = 1$$

and $1 \leqslant \operatorname{lub}(D) \leqslant \prod_{k=1}^n \operatorname{lub}(D_k)$, it follows that

$$1 \leqslant \operatorname{lub}(D) \leqslant \prod_{k=1}^{n} \operatorname{lub}(D_k) = 1^n = 1.$$

Therefore

$$\operatorname{lub}(D) = 1,$$

i.e., $\|\cdot\|$ is monotonic. ■

III. EXAMPLES AND CONCLUSIONS

By giving a set of examples, first we show that the inclusions in Lemma 2.2 are strict, and then by using these examples we give a characterization of classes of norms discussed previously with respect to the interval $[-1,1]$.

For $s \geqslant 0$ and $r_0 \in [0,1]$, let $\|\cdot\|_{s,r_0}$ be the following norm on $\mathbb{R}^n$ $(n \geqslant 2)$:

$$\left\|(x_1, x_2, \dots, x_n)^T\right\|_{s,r_0} = \max_{1 \leqslant i \leqslant n} \left\{ |(n+s+r_0-1)x_i|, \left| sx_i + \sum_{j=1}^{n} x_j \right| \right\}.$$

We claim that $\|\cdot\|_{s,r_0}$ satisfies the following condition:

$$\forall k = 1,2,\dots,n, \quad \operatorname{lub}(H_k(r_1)) = 1 \text{ and } \operatorname{lub}(H_k(r_2)) > 1,$$

where $r_1 \in [-1, r]$ and $r_2 \in (r,1]$. To prove this, let

$$M = \left\{ (x_1, x_2, \dots, x_n);\ \max_{1 \leqslant i \leqslant n} |x_i| = 1 \right\}$$

and let

$$u_1 = (-1,1,1,\dots,1), \qquad u_2 = (1,-1,1,\dots,1), \dots, \qquad u_n = (1,1,\dots,-1).$$

Then for $r_1 \in [-1, r_0]$,

$$\operatorname{lub}(H_k(r_1)) = \max_{v \in M} \frac{\|H_k(r_1)v\|_{s,r_0}}{\|v\|_{s,r_0}} \qquad (k = 1,2,\dots,n);$$

if $v = (x_1, x_2, \dots, x_n)^T$, then

$$\begin{aligned} H_k(r_1)v &= H_k(r_1)(x_1, x_2, \dots, x_n)^T = (x_1, x_2, \dots, r_1 x_k, x_{k+1}, \dots, x_n) \\ &= (y_1, y_2, \dots, y_n)^T; \end{aligned}$$

so $\|H_k(r_1)v\|_{s,r_0} = \|(y_1, y_2, \dots, y_n)^T\|_{s,r_0}$, and we have $\max_{1 \leqslant i \leqslant n}\{|y_i|\} \leqslant \max_{1 \leqslant i \leqslant n}\{|x_i|\} = 1$ and $\max_{1 \leqslant i \leqslant n}\{|sy_i + \sum_{j=1}^n y_j|\} \leqslant n+s+r_1-1 \leqslant n+s+r_0-1 = \|v\|_{s,r_0}$; hence $\|H_k(r_1)v\|_{s,r_0} \leqslant \|v\|_{s,r_0}$. Clearly for $1 \leqslant j \leqslant n$, $\|H_k(r_1)u_j\|_{s,r_0} = \|u_j\|_{s,r_0} = n+s+r_0-1$. Thus $\operatorname{lub}(H_k(r_1)) = 1$. If $r_2 \in (r_0, 1]$, then $\|H_k(r_2)u_k\|_{s,r_0} = n+s+r_2-1 > n+s+r_0-1 = \|u_j\|_{s,r_0}$, so $\operatorname{lub}(H_k(r_2))$

$\geqslant (n+s+r_2-1)/(n+s+r_0-1)>1$. Now it is clear that the inclusions in Lemma 2.2 are strict. We call r_0 the *radius* of *monotonicity* of the norm $\|\cdot\|_{s,r_0}$. Thus $\mathfrak{N}_{r_0}$ is the class of real vector norms of radius of monotonicity r_0, i.e., if $\|\cdot\|\in\mathfrak{N}_{r_0}$ and $r\in[-1,1]$, then

$$\operatorname{lub}(H_k(r))\begin{Bmatrix} =1 & \text{if} \quad -1\leqslant r\leqslant r_0 \\ >1 & \text{otherwise} \end{Bmatrix} \qquad (k=1,2,\ldots,n).$$

Combining the previous statements, the condition (1.13), and the equivalence of the conditions (1.8), (1.10), and (1.14), finally we may conclude that for $r_0\in[0,1]$, the class $\mathfrak{N}_{r_0}$ consists of norms on $\mathbb{R}^n$, satisfying

$$\|(x_1,x_2,\ldots,x_n)^T\|\leqslant\|(y_1,y_2,\ldots,y_n)^T\|$$

provided $|x_i|\leqslant|y_i|$ whenever $x_iy_i\geqslant 0$, and $|x_j|\leqslant r_0|y_j|$ whenever $x_jy_j<0$.

REFERENCES

1 F. L. Bauer, J. Stoer, and C. Witzgall, Absolute and monotonic norms, *Numer. Math.* 3:257–264 (1961).

2 R. E. Funderlic, Some characterizations of orthant monotonic norms, *Linear Algebra Appl.* 28:77–83 (1979).

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